Appendix D

Total Escape Probability for Chain Graphs, $\Sigma_\beta^{C_N}$

So long as there is at least one escape route from C_N the total escape probability must be unity:

$$\Sigma_\beta^{C_N} = \sum_{j=1}^{N} \mathcal{E}_j^{C_N} \mathcal{S}_{j,\beta}^{C_N} = 1, \tag{D.1}$$

otherwise, if $Adj[C_N]$ is the empty set, we have

$$\Sigma_\beta^{C_N} = \sum_{j=1}^{N} \mathcal{S}_{j,\beta}^{C_N} = 1. \tag{D.2}$$

For example, to show that the formulae in Appendix C are consistent with the first result we expand

$$\mathcal{E}_j^{C_N} = 1 - P_{j-1,j} - P_{j+1,j} \tag{D.3}$$

and define

$$P_{0,1} = P_{N+1,N} = 0 \tag{D.4}$$

for convenience, so that

$$\begin{aligned}
\Sigma_\beta^{C_N} &= \mathcal{S}_{1,\beta}^{C_N} - P_{0,1}\mathcal{S}_{1,\beta}^{C_N} - P_{2,1}\mathcal{S}_{1,\beta}^{C_N} \\
&+ \mathcal{S}_{2,\beta}^{C_N} - P_{1,2}\mathcal{S}_{2,\beta}^{C_N} - P_{3,2}\mathcal{S}_{2,\beta}^{C_N} \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
&+ \mathcal{S}_{N-1,\beta}^{C_N} - P_{N-2,N-1}\mathcal{S}_{N-1,\beta}^{C_N} - P_{N,N-1}\mathcal{S}_{N-1,\beta}^{C_N} \\
&+ \mathcal{S}_{N,\beta}^{C_N} - P_{N-1,N}\mathcal{S}_{N,\beta}^{C_N} - P_{N+1,N}\mathcal{S}_{N,\beta}^{C_N}.
\end{aligned} \tag{D.5}$$

Using the recursion relations in Equation C.2 (which assume that there is an escape route from C_N) we can show that

$$\mathcal{S}^{C_N}_{j,\beta} - \mathcal{S}^{C_N}_{j-1,\beta} P_{j,j-1} - \mathcal{S}^{C_N}_{j+1,\beta} P_{j,j+1} = 0, \tag{D.6}$$

for $j \neq \beta$. We can now group together terms in Equation D.5 into sets of three that sum to zero. The terms that do not immediately cancel are as follows. From the first and second lines of Equation D.5 we have

$$\mathcal{S}^{C_N}_{1,\beta} - P_{2,1} \mathcal{S}^{C_N}_{2,\beta} = 0 \tag{D.7}$$

because

$$\mathcal{S}^{C_N}_{1,\beta} = \mathcal{S}^{C_N}_{2,\beta} P_{2,1} L_1 = \mathcal{S}^{C_N}_{2,\beta} P_{2,1}. \tag{D.8}$$

Similarly, on the last two lines we find

$$\mathcal{S}^{C_N}_{N,\beta} - P_{N,N-1} \mathcal{S}^{C_N}_{N-1,\beta} = 0 \tag{D.9}$$

because

$$\mathcal{S}^{C_N}_{N,\beta} = P_{N,N-1} \mathcal{S}^{C_N}_{N-1,\beta} R_N = P_{N,N-1} \mathcal{S}^{C_N}_{N-1,\beta}. \tag{D.10}$$

The final remaining terms are:

$$\begin{aligned} & \mathcal{S}^{C_N}_{\beta,\beta} - \mathcal{S}^{C_N}_{\beta-1,\beta} P_{\beta,\beta-1} - \mathcal{S}^{C_N}_{\beta+1,\beta} P_{\beta,\beta+1} \\ = \; & \mathcal{S}^{C_N}_{\beta,\beta} - \mathcal{S}^{C_N}_{\beta,\beta} \left(1 - \frac{1}{L_\beta}\right) - \mathcal{S}^{C_N}_{\beta,\beta} \left(1 - \frac{1}{R_\beta}\right) \\ = \; & \mathcal{S}^{C_N}_{\beta,\beta} \left(\frac{1}{L_\beta} + \frac{1}{R_\beta} - 1\right) \\ = \; & \mathcal{S}^{C_N}_{\beta,\beta} \left(\frac{L_\beta + R_\beta - L_\beta R_\beta}{L_\beta R_\beta}\right) \\ = \; & 1, \end{aligned} \tag{D.11}$$

which proves Equation D.1.